

Existence of piecewise weak solutions of a discrete Cucker-Smale's flocking model with a singular communication weight

Jan Peszek

Institute of Applied Mathematics and Mechanics, University of Warsaw,

ul. Banacha 2, 02-097 Warsaw, Poland

E-mail: j.peszek@mimuw.edu.pl

Abstract

We prove existence of global C^1 piecewise weak solutions for the discrete Cucker-Smale's flocking model with the communication weight

$$\psi(s) = s^{-\alpha}, 0 < \alpha < 1.$$

We also discuss the possibility of finite in time alignment of the velocities of the particles.

1 Introduction

We consider the Cucker-Smale's (C-S) flocking model describing a collective self-driven motion of self-propelled particles, which for some reason have a tendency to flock, such as flockings of birds, schoolings of fishes or concentration of network activity. The purpose of this paper is to prove existence of piecewise weak solutions for the C-S model with an unbounded and non-Lipschitz communication weight. In the classical, discrete C-S model we assume that we have exactly N particles in some unspecified space and that (x_i, v_i) denotes the position and velocity of i -th particle, thus for $i = 1, \dots, N$, we have

$$\frac{dx_i}{dt} = v_i. \tag{1.1}$$

Moreover we know that velocity of each particle changes in time according to the equation

$$\frac{dv_i}{dt} = \frac{1}{N} \sum_{j=1}^N \psi(|x_j - x_i|)(v_j - v_i), \quad (1.2)$$

where $\psi : [0, \infty) \rightarrow [0, \infty)$ is a given function called the *communication weight* or the *communication rate*. The communication weight may be interpreted as perception of the particles. Indeed, we have

$$\frac{1}{N} \sum_{j=1}^N \psi(|x_j - x_i|)(v_j - v_i) = \frac{1}{N} \sum_{j=1}^N \psi(|x_j - x_i|) \left[\frac{\sum_{j=1}^N \psi(|x_j - x_i|)v_j}{\sum_{j=1}^N \psi(|x_j - x_i|)} - v_i \right],$$

thus v_i evolves towards $\frac{\sum_{j=1}^N \psi(|x_j - x_i|)v_j}{\sum_{j=1}^N \psi(|x_j - x_i|)}$, which is a convex combination of velocities v_j and can be interpreted as the average velocity of the particles perceived by i -th particle. For instance if $\psi \equiv 1$ then it is exactly the average velocity of the particles and if $\psi \equiv 0$ then the particles move independently with constant velocity as if they did not see each other.

As the number of particles grows to infinity, i.e., $N \rightarrow \infty$, the discrete model is replaced by the following Vlasov-type equation:

$$\begin{aligned} \partial_t f + v \cdot \nabla f + \operatorname{div}_v(F(f)f) &= 0, \quad x \in \mathbb{R}^d, \quad v \in \mathbb{R}^d, \\ F(f)(x, v, t) &:= \int_{\mathbb{R}^{2d}} \psi(|y - x|)(w - v)f(y, w, t)dw dy, \end{aligned} \quad (1.3)$$

where $f = f(x, v, t)$ is a density of particles that at the time t have position x and velocity v .

1.1 Smooth communication weight

The most well known communication weight is a bounded and smooth one given by

$$\psi_{cs}(s) = \frac{K}{(1 + s^2)^{\frac{\beta}{2}}}, \quad \beta \geq 0, \quad K > 0, \quad (1.4)$$

or more generally ψ_{cs} - bounded and Lipschitz continuous. The C-S model with weight ψ_{cs} was introduced in 2007 by Cucker and Smale in [12] and was in some sense based on the paper by Viscek [36] from 1995 in which a model of flocking was introduced, such that each particle adjusted its velocity with respect to the average velocity of its neighbors. Since then existence, uniqueness, asymptotics and stability

for alignment models similar to C-S (both in continuous and discrete cases) were extensively studied, both from physical and biological point of view [13] -[16], [30], [33] - [35] and from more theoretical point of view [1, 2], [4] - [11], [17] - [25], [29, 31]. A nice and thorough study of the C-S flocking model with a bounded communication weight can be found in [28], where the interplay between discrete and continuous model is studied with measure valued solutions of the Vlasov type equation (1.3) or in [26], where the authors present a new, simple approach to the problem of existence and asymptotics. Recently other direction of studies was introduced - namely to couple a continuous C-S model treated as a Vlasov equation with Navier-Stokes system to model the motion of fluid imbeded particles. This approach is based on the paper by Beduin [32] and can be found in [3].

1.2 Singular communication weight

Our main interest is the C-S model with weight

$$\psi(s) = \begin{cases} s^{-\alpha} & \text{for } s > 0, \\ 0 & \text{for } s = 0, \end{cases} \quad \alpha \in (0, d). \quad (1.5)$$

With the loss of Lipschitz continuity of ψ the problem of existence and uniqueness for the discrete C-S model becomes more difficult. To our best knowledge there are no results in this direction, even though there are results on asymptotics in such case, see e.g. [26]. When dealing with the C-S model with bounded weight ψ_{cs} one makes use of Lipschitz continuity of ψ_{cs} as well as the structure of the model itself. As an example we will now present a simple application of the properties of the structure of our model. Namely we will prove that the average velocity of the particles

$$\bar{v}(t) := \frac{1}{N} \sum_{i=1}^N v_i(t)$$

is constant in time. Assuming that $x = (x_1, \dots, x_N)$ and $v = (v_1, \dots, v_N)$ is a sufficiently smooth solution of (1.1) and (1.2), we calculate the derivative of \bar{v} to get

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N v_i &= \frac{1}{N} \sum_{i,k=1}^N (v_k - v_i) \psi_n(|x_i - x_k|) = \\ \frac{1}{2N} \sum_{i,k=1}^N (v_k - v_i) \psi_n(|x_i - x_k|) &+ \frac{1}{2N} \sum_{i,k=1}^N (v_i - v_k) \psi_n(|x_i - x_k|) = 0, \end{aligned}$$

where the latter summand in the second line is obtained by substituting i and k . Clearly each such structure based property of the C-S model will remain true regardless of the

communication weight ψ as long as it is a nonnegative function. This is the first piece of information on which we base our hope to obtain some existence for C-S model with singular weight ψ . The second piece of information is that Lipschitz continuity and boundedness of ψ fails only at 0, which means that our main problem will be to prove existence in a neighborhood of each time t_0 at which some particles collide. However, heuristically in a neighborhood of each such point we have

$$x_i(t) - x_j(t) \approx t(v_i(t_0) - v_j(t_0)) \approx t(v_i(t) - v_j(t))$$

and since in (1.2) the function $t \mapsto \psi(|x_i(t) - x_j(t)|)$ comes always multiplied by $v_i(t) - v_j(t)$, we have

$$\left(\Psi(|x_i(t) - x_j(t)|)\right)' = \psi(|x_i(t) - x_j(t)|) \frac{(x_i(t) - x_j(t)) \cdot (v_i(t) - v_j(t))}{|x_i(t) - x_j(t)|} \approx \psi(|x_i(t) - x_j(t)|)(v_i(t) - v_j(t))$$

with $\Psi(s) := \frac{1}{1-\alpha} s^{1-\alpha}$ being a primitive of ψ , which is a Hölder continuous function, thus there is hope for some better regularity of v . These two observations were already used in [26] to obtain asymptotic flocking for C-S model with weight ψ . Occurrence of asymptotic flocking is a further clue that a C-S model with singular weight inherits some nice properties from the model with a smooth weight. Lastly in [26] existence for the discrete model served as a mean to obtain existence for the continuous model by defining the solution of the continuous model as a Wasserstein metric's limit of approximative discrete solutions. What is interesting from the point of view of the model with singular weight is that the existence of such limit was ensured by structure only, which means that it should work also in our case. Of course existence of the limit is not enough to prove that the limit actually satisfies any equation at all but it is a first step. In [26] it was the Lipschitz continuity of ψ_{cs} that served to prove that this limit was indeed a solution of the continuous C-S model, which indicates that this may be the most difficult part in case of the singular communication weight.

1.3 Preliminaries and notation

The definition of our piecewise weak solutions and the proof of their existence and regularity can be found in Chapter 2. In Chapter 3, we restrict ourselves to the case of two particles and present necessary and sufficient conditions on the initial data for the trajectories of the particles to stick together in a finite time.

Hereinafter $x = (x_1, \dots, x_N) \in \mathbb{R}^{Nd}$, where $x_i = (x_{i,1}, \dots, x_{i,d})$ denotes the position of the particles, $v = \dot{x}$ is their velocity, where N and d are the number of the particles and the dimension of the space. Approximate solutions x^n and v^n are defined in section 2.2. Moreover by $B_i(t)$ we will denote the set of all indexes j , such that the trajectory of x_j does not coincide with the trajectory of x_i as of the time t . Assuming that the trajectories, once coinciding cannot separate, we may define it as

$$B_i(t) := \{k = 1, \dots, N : x_k^n(t) \neq x_i^n(t) \text{ or } v_k^n(t) \neq v_i^n(t)\}, \quad (1.6)$$

since any two particles with sufficiently smooth trajectories have the same position and velocity at the time t , if and only if they move on the same trajectory. Further, by $W^{k,p}(\Omega)$ we denote the Sobolev space of the functions with up to k -th weak derivative belonging to the space $L^p(\Omega)$ and by $C_c^\infty(\Omega)$ we denote the space of smooth and compactly supported functions.

In the sequel we use the following lemmas, which are well known results from measure theory.

Lemma 1.1 (Vitali's convergence theorem). *Let $\Omega \subset \mathbb{R}^d$ be bounded and f_n be a sequence of functions converging a.e. to an a.e. finite function f . Then if f_n are uniformly integrable, then $f \in L^1(\Omega)$ and*

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu = 0.$$

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^d$ and $f_n, f, g_n, g : \Omega \rightarrow \mathbb{R}$ be measurable functions. If $f_n \rightarrow f$ a.e. in Ω , f_n is uniformly bounded in $L^\infty(\Omega)$ and $g_n \rightarrow g$ in $L^1(\Omega)$, then*

$$f_n g_n \rightarrow f g \text{ in } L^1(\Omega).$$

Lastly we will say that particles x_i and x_j *collide* at the time t if and only if $x_i(t) = x_j(t)$ but $x_i \neq x_j$ in some left-sided neighborhood of t and we will say that they *stick* at the time t if and only if they collide and $v_i(t) = v_j(t)$.

2 Main result

2.1 Outline

In this section we prove existence for the discrete C-S model (1.1) and (1.2) with a singular communication weight given by (1.5). Our strategy is based on the observation that the function $t \mapsto \psi(|x_i(t) - x_j(t)|)$ is Lipschitz continuous in a neighborhood of each time t_0 , such that for all i, j , we have $x_i(t_0) \neq x_j(t_0)$, which makes local existence in such points trivial. The idea is that if we can prove that the particles collide in some sense rarely, then the only difficulty will be to establish existence in a neighborhood of each point of collision of some particles. Technically we will obtain existence of solutions by approximating them with solutions of C-S model with bounded weights.

2.2 Approximate solutions

In this section we define the approximate solutions and present some of their most important properties. For each n let

$$\psi_n(s) = \begin{cases} \psi(s) & \text{if } s \geq (n-1)^{-\frac{1}{\alpha}}, \\ \text{smooth and monotone} & \text{if } n^{-\frac{1}{\alpha}} \leq s \leq (n-1)^{-\frac{1}{\alpha}}, \\ n & \text{if } s \leq n^{-\frac{1}{\alpha}} \end{cases}$$

for all $s \in [0, \infty)$ with ψ given by (1.5). For all n , functions ψ_n are smooth and bounded, thus C-S systems associated with these weights have unique solutions. This can be expressed by the following proposition.

Proposition 2.1. *For each positive integer n and for arbitrary initial data, the system*

$$\begin{cases} \dot{x}_i^n = v_i^n, \\ \dot{v}_i^n = \frac{1}{N} \sum_{k=1}^N (v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) \end{cases} \quad (2.1)$$

has a unique global classical solution x^n belonging to the class $(C^2([0, T]))^{Nd}$. Moreover, this solution is stable with respect to the initial data i.e. for all $x^{n,1}(0), x^{n,2}(0), v^{n,1}(0), v^{n,2}(0)$, we have

$$\|x^{n,1} - x^{n,2}\|_{W^{1,\infty}([0,T])} < L(n) \left(|x^{n,1}(0) - x^{n,2}(0)| + |v^{n,1}(0) - v^{n,2}(0)| \right),$$

where $L(n)$ is a constant depending on n (and also on T, d, α and N).

The proof of this proposition is standard and we omit it.

Remark 2.1. Stability with respect to the initial data is uniform in a neighborhood of each time in which no collision occurs. Indeed, if in some interval $[0, t]$ we have

$$\inf_{i,j} |x_i^n - x_j^n| > \delta$$

for all n , then C-S systems associated with ψ_n are exactly the same for all $n > \delta^{-\alpha} + 1$. It follows from the fact that for $n > \delta^{-\alpha} + 1$ and $s \geq \delta$ all functions ψ_n coincide. It implies that if we consider functions x^n restricted to $[0, t]$, then Lipschitz continuity with respect to the initial data mentioned in Proposition 2.1 holds with an n independent constant $L(\lfloor \delta^{-\alpha} \rfloor + 2)$ which corresponds to the C-S system associated with the smallest weight for which ψ_n coincide for $s \geq \delta$.

Next, let us state some properties of solutions of the C-S model associated with ψ_n .

Proposition 2.2. *Let x^n be a solution of the C-S model associated with weight ψ_n . Then x^n has the following properties:*

1. It belongs to the class C^∞ in a neighborhood of every such point t , that $|x_i^n(t) - x_j^n(t)| > 0$ for all $i, j = 1, \dots, N$.

2. The average velocity of the particles is constant:

$$\frac{1}{N} \sum_{i=1}^N v_i(t) = \text{const.}$$

3. Velocity v^n is bounded: there exists a constant $M(n)$ such that for all $i = 1, \dots, N$, we have

$$\|v_i^n\|_{L_\infty([0, T])} \leq M(n).$$

4. If the initial data $x^n(0), v^n(0)$ are uniformly bounded, then also v^n is uniformly bounded: there exists a constant M such that for all $i = 1, \dots, N$ and all $n = 1, 2, \dots$, we have

$$\|v_i^n\|_{L_\infty([0, T])} \leq M.$$

5. Acceleration \dot{v}_i^n is bounded by $2M(n)n$.

6. If at some point t we have $x_i^n(t) = x_j^n(t)$ and $v_i^n(t) = v_j^n(t)$ for any $i, j = 1, \dots, N$, then $x_i^n \equiv x_j^n$ on $[t, T]$.

7. If at some point t we have $v_i^n(t) = v_j^n(t)$ for all $i, j = 1, \dots, N$, then v^n is constant on $[t, T]$.

Proof. 1. Since x^n is continuous, if at some point t all the particles have different positions i.e. $|x_i^n(t) - x_j^n(t)| > 0$ for all $i, j = 1, \dots, N$ then it is also true in some neighborhood of t . Moreover in this neighborhood of t the right-hand side of (2.1)₂ is differentiable, which by iteration implies that x^n is smooth at t .

2. This part was already done in section 1.2.

3. Let $r_n(t) := \sum_{i,j=1}^N (v_i^n(t) - v_j^n(t))^2$. By (2.1)₂, we have

$$\begin{aligned} r'_n &= 2 \sum_{i,j=1}^N (v_i^n - v_j^n) \left(\frac{1}{N} \sum_{k=1}^N (v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) - \frac{1}{N} \sum_{k=1}^N (v_k^n - v_j^n) \psi_n(|x_j^n - x_k^n|) \right) = \\ &= \frac{2}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n) (v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) - \frac{2}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n) (v_k^n - v_j^n) \psi_n(|x_j^n - x_k^n|). \end{aligned}$$

Again, we substitute i and k in the first summand and j and k in the second summand to obtain

$$\begin{aligned}
r'_n &= \frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n)(v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) + \frac{1}{N} \sum_{i,j,k=1}^N (v_k^n - v_j^n)(v_i^n - v_k^n) \psi_n(|x_i^n - x_k^n|) \\
&\quad - \frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_j^n)(v_k^n - v_j^n) \psi_n(|x_j^n - x_k^n|) - \frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_k^n)(v_j^n - v_k^n) \psi_n(|x_j^n - x_k^n|) = \\
&\quad - \frac{1}{N} \sum_{i,j,k=1}^N (v_i^n - v_k^n)^2 \psi_n(|x_i^n - x_k^n|) - \frac{1}{N} \sum_{i,j,k=1}^N (v_j^n - v_k^n)^2 \psi_n(|x_j^n - x_k^n|) = \\
&\quad = -2 \sum_{i,j=1}^N (v_i^n - v_j^n)^2 \psi_n(|x_i^n - x_j^n|) \leq 0.
\end{aligned}$$

Thus for each n , function r_n is nonincreasing with it's maximum at 0 i.e. $r_n(t) \leq r_n(0)$. Now let \bar{v}^n be the average velocity, which as we know from property 2 is a constant. We have

$$\sum_{i=1}^N (\bar{v}^n - v_i^n)^2 = \sum_{i=1}^N \left(\frac{1}{N} \sum_{j=1}^N v_j^n - v_i^n \right)^2 = \frac{1}{N^2} \sum_{i=1}^N \left(\sum_{j=1}^N (v_j^n - v_i^n) \right)^2 \leq \frac{1}{N} \sum_{i,j=1}^N (v_j^n - v_i^n)^2 = \frac{1}{N} r_n(0).$$

Lastly we have

$$|v_i^n| \leq |v_i^n - \bar{v}^n| + |\bar{v}^n| \leq C(N) \sqrt{r_n(0)} + |\bar{v}^n| \leq C(N) \sqrt{r_n(0)} =: M(n),$$

where $C(N)$ is a generic constant depending on N .

4. We simply note that if initial velocity is uniformly bounded, then $M(n) \leq M$ for some M independent of n .

Point 5 follows immediately from property 3 and equation (2.1)₂, while points 6 and 7 are obvious consequences of uniqueness of the solutions. \square

Remark 2.2. Property 6 from the above proposition implies that the acceleration equation (2.1)₂ can be replaced by:

$$\dot{v}_i^n = \frac{1}{N} \sum_{k \in B_i(t)} (v_k^n - v_i^n) \psi_n(|x_k^n - x_i^n|), \tag{2.2}$$

where $B_i(t)$ is defined by (1.6), with

$$\dot{v}_i^n = 0$$

should set $B_i(t)$ be empty. This technical observation will be useful later on.

Hereinafter we will use $M(n)$ and M in the same roles as in Proposition 2.2. We end this section with an important lemma that is in fact our way to deal with existence in a right sided neighborhood of a point of collision.

Lemma 2.1. *Let x^n be a solution of C-S system on the time interval $[0, T]$ with weight ψ_n and initial data $x(0), v(0)$ – independent of n . Then there exists an interval $[0, t]$, such that all velocities v^n are uniformly Hölder continuous on $[0, t]$.*

To prove this lemma we need yet another, technical lemma.

Lemma 2.2. *If $x_i(0) = x_j(0)$, then for all n , there exists an interval $(0, t_n]$, such that*

$$|v_i^n(s) - v_j^n(s)| \leq 4 \frac{|(v_i^n(s) - v_j^n(s)) \cdot (x_i^n(s) - x_j^n(s))|}{|x_i^n(s) - x_j^n(s)|}$$

for $s \in (0, t_n]$.

Proof. By property 5 from Proposition 2.2, we have

$$v_i^n(s) - v_j^n(s) = v_i^n(0) - v_j^n(0) + r_n(s), \quad (2.3)$$

where $|r_n(s)| \leq 2|s|Mn$. Moreover as $x_i^n - x_j^n$ is a C^2 function, by Taylor's formula

$$x_i^n(s) - x_j^n(s) = s \cdot (v_i^n(0) - v_j^n(0)) + o_n(s) = s(v_i^n(s) - v_j^n(s) - r_n(s)) + o_n(s), \quad (2.4)$$

where

$$o_n(s) := \int_0^s (\ddot{v}_i^n - \ddot{v}_j^n)(s - \theta) d\theta, \quad |o_n(s)| \leq 2|s|^2 Mn.$$

Thus

$$\begin{aligned} |(v_i^n(s) - v_j^n(s))(x_i^n(s) - x_j^n(s))| &= |s(v_i^n(s) - v_j^n(s))^2 - s(v_i^n(s) - v_j^n(s))r_n(s) + (v_i^n(s) - v_j^n(s))o_n(s)| \geq \\ &\geq s(v_i^n(s) - v_j^n(s))^2 - s|(v_i^n(s) - v_j^n(s))r_n(s)| - |(v_i^n(s) - v_j^n(s))o_n(s)| \geq \frac{s}{2}(v_i^n(s) - v_j^n(s))^2 \end{aligned} \quad (2.5)$$

assuming that $s \in (0, t_n]$, where t_n is the supremum of all times s_n , such that for all $s \in (t, s_n]$, we have

$$|r_n(s)| \leq \frac{1}{4}|v_i^n(s) - v_j^n(s)|, \quad |o_n(s)| \leq \frac{s}{4}|v_i^n(s) - v_j^n(s)|. \quad (2.6)$$

To check that $t_n > 0$, we notice that for

$$s_n := \frac{|v_i^n(0) - v_j^n(0)|}{10M(n)n} \quad (2.7)$$

and $s \in [0, s_n]$, we have

$$|r_n(s)| \leq \frac{1}{5}|v_i^n(0) - v_j^n(0)|, \quad |o_n(s)| \leq \frac{s}{5}|v_i^n(0) - v_j^n(0)|, \quad (2.8)$$

which together with (2.3) implies that

$$\frac{4}{5}|v_i^n(0) - v_j^n(0)| \leq |v_i^n(s) - v_j^n(s)|,$$

condition (2.6) is satisfied. Therefore by taking s_n given by (2.7) we get (2.5). Now by (2.3) and (2.4) on $(0, s_n]$ we also have

$$|x_i^n(s) - x_j^n(s)| \leq 2s|v_i^n(s) - v_j^n(s)|, \quad (2.9)$$

which together with (2.5) proves that there exists $s_n > 0$ such that on $(0, s_n]$ the assertion holds. Now we define t_n as the supremum of all such times s_n . This finishes the proof. \square

Next we can proceed with the proof of Lemma 2.1.

Proof of Lemma 2.1. The proof will follow by 2 steps. In step 1 we prove that for each n there exists an interval $[0, t_n]$ on which v^n is Hölder continuous with a constant independent of n , while in step 2 we establish a lower bound on t_n that is independent of n .

STEP 1. It suffices to show the assertion separately for all particles, thus let us fix $i = 1, \dots, N$ and consider x_i . By Remark 2.2 for all s , we have

$$\begin{aligned} |v_i^n(s) - v_i^n(0)| &= \left| \int_0^s \dot{v}_i^n(\theta) d\theta \right| \leq \frac{1}{N} \sum_{k \in B_i(0)} \int_0^s |v_k^n - v_i^n| \psi_n(|x_k^n - x_i^n|) d\theta = \\ &= \frac{1}{N} \sum_{k \in B_i^0} \int_0^s |v_k^n - v_i^n| \psi_n(|x_k^n - x_i^n|) d\theta + \frac{1}{N} \sum_{k \in B_i^+} \int_0^s |v_k^n - v_i^n| \psi_n(|x_k^n - x_i^n|) d\theta =: I + II, \end{aligned}$$

where

$$B_i^0 := \{j \in B_i(0) : |x_j(0) - x_i(0)| = 0\}, \quad B_i^+ := \{j \in B_i(0) : |x_j(0) - x_i(0)| > 0\}$$

and $B_i(0)$ is the defined by (1.6) set of all particles that have different trajectories than x_i . Thus B_i^0 consists of all particles that start from the same position as x_i but with different velocities, while B_i^+ consists of all particles that start from a different position than x_i . We may assume that $B_i(0)$ is not empty as otherwise all x^n are constantly equal $x^n(0)$ and the assertion holds. Thus at least one of sets B_i^0 or B_i^+ is nonempty. Now we estimate I and II separately starting with I . For $j \in B_i^0$, we have $|v_j^n(0) - v_i^n(0)| > 0$

and by its continuity there exists t_n such that $|x_j^n(s) - x_i^n(s)| > 0$ and consequently $\psi_n(|x_j^n(s) - x_i^n(s)|) \leq \psi(|x_j^n(s) - x_i^n(s)|)$ in $(0, t_n]$. Together with Lemma 2.2 it implies that

$$I \leq \frac{4}{N} \sum_{j \in B_i^0} \int_0^s \frac{|(v_j^n - v_i^n) \cdot (x_j^n - x_i^n)|}{|x_j^n - x_i^n|} \psi(|x_j^n - x_i^n|) d\theta.$$

We claim that, since $\Psi(|x_j^n(0) - x_i^n(0)|) = 0$ for all $j \in B_i^0$, then

$$\int_0^s \frac{|(v_j^n - v_i^n) \cdot (x_j^n - x_i^n)|}{|x_j^n - x_i^n|} \psi(|x_j^n - x_i^n|) d\theta = \Psi(|x_j^n(s) - x_i^n(s)|),$$

where $\Psi(s) = \frac{1}{1-\alpha} s^{1-\alpha}$ is a primitive of ψ . Indeed, we have

$$\Psi(|x_j^n(s) - x_i^n(s)|) = \int_0^s \Psi'(|x_j^n - x_i^n|) d\theta \leq \int_0^s \psi(|x_j^n - x_i^n|) \frac{|(x_j^n - x_i^n)(v_j^n - v_i^n)|}{|x_j^n - x_i^n|} d\theta$$

and since $\psi \geq 0$ we can substitute the above inequality with an equality provided that on $(0, t_n]$ the function $\xi(s) := (x_j^n(s) - x_i^n(s))(v_j^n(s) - v_i^n(s))$ has a constant sign. To prove that ξ has a constant sign it suffices to show that $|\xi| > 0$ in $(0, t_n]$, which is an immediate consequence of Lemma 2.2. Thus we proved that

$$I \leq \frac{4}{N} \sum_{j \in B_i^0} \Psi(|x_j^n(s) - x_i^n(s)|) = \frac{4}{N(1-\alpha)} \sum_{j \in B_i^0} \left| (x_j^n(s) - x_i^n(s)) \right|^{1-\alpha} \leq \frac{4M^{1-\alpha}}{N(1-\alpha)} \sum_{j \in B_i^0} |s|^{1-\alpha} \leq \frac{4M^{1-\alpha}}{1-\alpha} |s|^{1-\alpha},$$

where we use inequality $|x_j^n(s) - x_i^n(s)| \leq M|s|$ that follows from property 4 from Proposition 2.2. To estimate II we first notice that since for all $j \in B_i^+$, we have $|x_j^n(0) - x_i^n(0)| > 0$ then there exists $\delta > 0$ such that $|x_j^n(0) - x_i^n(0)| > \delta$ for all $j \in B_i^+$. Then, by property 4 from Proposition 2.2 there exists an n independent interval $[0, t_0]$ on which $|x_j^n - x_i^n| > \delta$ for all $j \in B_i^+$. On this interval

$$\psi_n(|x_j^n(s) - x_i^n(s)|) \leq \delta^{-\alpha}.$$

Therefore

$$II \leq \frac{1}{N} \sum_{j \in B_i^+} 2|s|M\delta^{-\alpha} \leq 2t_0^\alpha M\delta^{-\alpha} |s|^{1-\alpha}$$

and adding our estimations of I and II we get

$$|v_i^n(s) - v_i^n(0)| \leq L|s|^{1-\alpha}$$

with $L = \frac{4M^{1-\alpha}}{1-\alpha} + 2t_0^\alpha M\delta^{-\alpha}$ on interval $[0, t_n] \cap [0, t_0]$. For simplicity let us denote $\min\{t_n, t_0\}$ again by t_n . This finishes step 1.

STEP 2. In step 1 we proved that for each n there exists an interval $[0, t_n]$ in which v_i^n is Hölder continuous with a constant independent of n . Now we prove that there exists $t > 0$, such that for all n , we have $t \leq t_n$ and thus in $[0, t]$ all functions v_i^n are uniformly Hölder continuous. There are exactly 3 instances, when we bound t_n from the above:

1. In the proof of Lemma 2.2.
2. While ensuring that for all $k \in B_i^0$ we have $|v_k^n - v_i^n| > 0$ in $[0, t_n]$.
3. While ensuring that for all $k \in B_i^0$ the function ξ is positive in $(0, t_n]$.

If each of these bounds from above can be bounded from below by a constant independent of n , then so can be t_n for all n .

1. In Lemma 2.2, t_n was the supremum of all times s_n , such that for all $s \in (0, t_n]$ conditions (2.6) and (2.9) are satisfied. However from step 1 we may estimate t_n better than we could in the proof of Lemma 2.2. We have

$$|r_n(s)| \leq 2L|s|^{1-\alpha} \quad \text{and} \quad |o_n(s)| \leq 2L|s|^{2-\alpha},$$

thus by taking

$$t_0 := \left(\frac{1}{10L} |v_k^n(0) - v_i^n(0)| \right)^{\frac{1}{1-\alpha}} \quad (2.10)$$

we ensure that (2.8) and consequently (2.6) is satisfied. With the same t_0 we obtain also condition (2.9).

2. For $k \in B_i^0$ we have $|v_k^n(0) - v_i^n(0)| > 0$, thus

$$|v_k^n(s) - v_i^n(s)| \geq |v_k^n(0) - v_i^n(0)| - 2L|s|^{1-\alpha},$$

which is positive for $s \leq t_0$ with t_0 defined by (2.10).

3. To prove that ξ has a constant sign in $[0, t_n]$ we applied Lemma 2.2 concluding that $|\xi(s)|$ is positive, provided that s belongs to the interval on which the thesis of Lemma 2.2 holds and we proved above that this interval includes $(0, t_0]$.

Therefore all bounds from points 1,2 and 3 are satisfied for t_0 defined by (2.10) and it is clearly n -independent. Thus we proved that there exists an interval $[0, t]$ with $t \geq t_0$ in which all functions v_i^n are uniformly Hölder continuous. \square

2.3 Definition of the solution

Before we define the solution let us recall property 6 from Proposition 2.2, which basically states that the trajectories of the particles cannot separate if they stick together at some point. This is an obvious consequence of the uniqueness for the approximate solutions. However, since ψ is singular at 0 it may happen that the solutions of the (C-S) model with ψ are not unique and that the trajectories may split as in the case of the well known example $\dot{y} = cx^{\frac{1}{3}}$. In fact a loss of uniqueness may happen at each time t , such that there exist i and j , such that $x_i(t) = x_j(t)$. It is problematic because such times t include not only each time of a collision but also each time at which some particles are stuck together. Thus if for example two particles x_i and x_j start with the same position and velocity, then we may lose uniqueness at an arbitrary time $t > 0$. Therefore we will enforce that the once stuck trajectories cannot separate. We will do this by replacing equation (1.2) with (2.2), which does not distinguish trajectories that once stuck together. Hereinafter we consider (C-S) model defined by (1.1) and (2.2). For this model we still do not have uniqueness but the times at which we lose it are restricted only to the times of collisions, which as we will prove occur in some sense rarely.

Thus our problem and it's solution is defined as follows.

Definition 2.1. Let $\{T_n\}_{n \in \mathbb{N}} \cup \{0\}$ be the set of all times of collision of some particles and for each n let $0 < T_n \leq T_{n+1}$. For $n \geq -1$, on each interval $[T_n, T_{n+1}]$ (we assume that $T_{-1} = 0$) we consider the problem

$$\begin{cases} \frac{dx_i}{dt} = v_i, \\ \frac{dv_i}{dt} = \frac{1}{N} \sum_{k \in B_i(T_n)} (v_k^n - v_i^n) \psi_n(|x_k^n - x_i^n|), \end{cases} \quad (2.11)$$

for $t \in [T_n, T_{n+1}]$, with initial data $x(T_n), v(T_n)$.

We say that x solves (2.11) on the time interval $[0, T]$, with weight given by (1.5) and arbitrary initial data $x(0), v(0)$ if and only if for all T_n and all $t \in (T_n, T_{n+1})$ the function $x \in (C^1([0, T]))^{Nd}$ is a weak in $(W^{2,1}([T_n, t]))^{Nd}$ solution of (2.11), the initial data are correct (i.e. $x(0) = x(T_{-1})$ and $v(0) = v(T_{-1})$) and for some n , we have $T \leq T_n$.

This definition may not be clear at the first glance. It is somewhat weaker than a weak solution but stronger than an a.e. solution. Such definition is based on the idea described in section 2.1: the solution exists in a weak sense between two collision times T_{n-1} and T_n . However as it approaches T_n , the second derivative of x may blow up. Despite this v is still continuous in a left sided neighborhood of T_n and has a limit at T_n^- . Therefore we may continuously define it at T_n ensuring existence of unique initial data for local weak existence in $[T_n, T_{n+1})$.

2.4 Existence up to the time of collision

Before we begin let us state the following simple remark.

Remark 2.3. Property 4 from Proposition 2.2 implies equicontinuity of x^n , thus by Arzela-Ascoli theorem there exists a $(C([0, T]))^{Nd}$ convergent subsequence x^{n_k} . From this point we pick one of such convergent subsequences and aim to prove that it has a $(C^1([0, T]))^{Nd}$ convergent subsequence. For simplicity of notation we will assume that $x^n = x^{n_k}$.

In this section we will prove that the approximate solutions converge in every interval $[0, t] \subset [0, T_0)$, where T_0 is the time of the first collision of the particles. Additionally we will prove that their limit is a weak solution in $(W^{2,1}([0, t]))^{Nd}$. Let us begin with defining T_0 by means of the approximate solutions:

$$T_0 := \inf\{t > 0 : \min_{\substack{i=1, \dots, N \\ j \in B_i(0)}} \lim_{n \rightarrow \infty} |x_i^n - x_j^n| = 0\}.$$

Note that the limit in the above definition exists, since we are restricted to a $(C([0, T]))^{Nd}$ convergent subsequence.

Remark 2.4. Clearly if $t < T_0$ then there exists $\delta > 0$, such that

$$\min_{\substack{i=1, \dots, N \\ j \in B_i(0)}} \lim_{n \rightarrow \infty} |x_i^n - x_j^n| > \delta,$$

which further implies that for all i, j , there exists n_0 such that for all $n > n_0$, we have $|x_i^n - x_j^n| > \delta$. On the other hand

$$\lim_{n \rightarrow \infty} |x_i^n(T_0) - x_j^n(T_0)| = 0$$

and assuming that x is a $(C([0, T_0]))^{Nd}$ limit of x^n , we have $x_i(T_0) = x_j(T_0)$, which means that T_0 is a point of collision for x .

Proposition 2.3. For $n = 1, 2, \dots$ let x^n be a solution to C-S system on interval $[0, T]$ with weight ψ_n and an independent of n initial data $x(0)$ and $v(0)$. There exists an interval $[0, T_0)$ such that for any $[0, t] \subset [0, T_0)$ solutions x^n have a subsequence that converges in $(C^1([0, t]))^{Nd}$.

Proof. If for all $i, j = 1, \dots, N$ initial velocity $v_i(0) = v_j(0)$ then by property 7 from Proposition 2.2, we have $v^n \equiv v(0)$ for all n and the assertion holds with $T_0 = T$. From this point we assume that there exist $i, j = 1, \dots, N$ such that $v_i(0) \neq v_j(0)$. Recall $B_i(0)$ defined by (1.6) – the set of all indexes which are directly involved in the evolution of v_i^n – from this point we will only consider $j \in B_i(0)$. There are two possibilities:

(A) For all i and $j \in B_i(0)$ we have $x_j(0) \neq x_i(0)$.

(B) There exists i and $j \in B_i(0)$ such that $x_j(0) = x_i(0)$.

(A) In this case there exists δ such that for all $i = 1, \dots, N$ and all $j \in B_i(0)$ we have $|x_i(0) - x_j(0)| > \delta$ and by Remark 2.4 for all $t < T_0$ there exist $\delta_t \in (0, \delta]$, such that for all i, j we have $|x_i - x_j| > \delta_t$ on $[0, t]$, which implies that $\psi(|x_i - x_j|) \leq \delta^{-\alpha}$ and all velocities v^n are uniformly Lipschitz continuous on $[0, t]$. Therefore by Arzela-Ascoli theorem there exists a $(C^1([0, t]))^{Nd}$ convergent subsequence of x^n .

(B) In the second case, there exist i and j , such that $x_i(0) = x_j(0)$ and $v_i(0) \neq v_j(0)$ and we may not proceed as in case **(A)** as there is no such $\delta > 0$, that $|x_i(0) - x_j(0)| > \delta$. However for this situation we have prepared Lemma 2.1, which implies uniform Hölder continuity of v^n in some neighborhood of 0. Therefore for sufficiently small t_0 and $j \in B_i(0)$, such that $x_j(0) = x_i(0)$, we have

$$|x_i^n(s) - x_j^n(s)| \geq s(|v_i(0) - v_j(0)| - 2Ls^{1-\alpha}) \geq s\frac{1}{2}|v_i(0) - v_j(0)| =: \delta_s > 0$$

for $s \in [0, t_0]$. On the other hand, for all $j \in B_i(0)$ such that $x_j(0) \neq x_i(0)$, from property 4 of Proposition 2.2, we have

$$|x_i^n(s) - x_j^n(s)| \geq |x_i(0) - x_j(0)| - 2Ms \geq \delta_s$$

for all $n = 1, 2, \dots$ and all $s \in [0, t_1]$ with $0 < t_1 < 1$ possibly smaller than t_0 . Thus in t_1 we end up in a situation from case **(A)** with

$$|x_i^n(t_1) - x_j^n(t_1)| \geq \delta_{t_1}$$

and all velocities v^n are uniformly Hölder continuous on $[0, t_1]$ and uniformly Lipschitz continuous on $[t_1, t]$ for all $t < T_0$. Again by Arzela-Ascoli theorem, there exists a $(C^1([0, t]))^{Nd}$ convergent subsequence of x^n . \square

Remark 2.5. As in Remark 2.3, even though x from Proposition 2.3 is a limit of some subsequence of x^n , we will assume that it is in fact a limit of the whole sequence x^n (by restricting the approximate solutions to only those, which approximate x). Such assumption will pose no threat to our reasonings as long as they will not involve uniqueness of x .

Corollary 2.1. *Let x be as in Remark 2.5. Then x is a local classical solution to C-S system in the interval $(0, T_0)$. Moreover*

1. *For all $i, j = 1, \dots, N$, we have $|x_j - x_i| > 0$ in $(0, T_0)$.*
2. *The function x is smooth in $(0, T_0)$.*

Proof. By the definition of T_0 we get assertion 1, which on the other hand implies that in a neighborhood of each $t \in (0, T_0)$ all the derivatives of x^n are uniformly bounded, which by Arzela–Ascoli theorem implies that x is smooth in $(0, T_0)$. With this, to prove that x solves C-S system with weight ψ , it suffices to take a $(C^2([t - \epsilon, t + \epsilon]))^{Nd}$ limit of systems associated with weights ψ_n , with $[t - \epsilon, t + \epsilon] \subset (0, T_0)$. \square

Our next step is to show that the function x actually satisfies our problem in a weak sense in every interval $[0, t] \subset [0, T_0)$ (though to prove that it satisfies Definition 2.1 we still need continuity of v at T_0).

Proposition 2.4. *For all $t \in [0, T_0]$ the function x is a weak $(W^{2,1}([0, t]))^{Nd}$ solution of (2.11).*

Proof. From Proposition 2.3 and Corollary 2.1 we know that $x \in (C^1([0, T_0]))^{Nd}$ and that T_0 is the time of the first collision of the particles. It suffices to show that x satisfies (2.11) weakly in intervals $[0, t]$ for $t \nearrow T_0$. Since x^n satisfy (2.11)₁ and $x^n \rightarrow x$ in $(C^1([0, t]))^{Nd}$, then x satisfies (2.11)₁ with $v = \lim_{n \rightarrow \infty} v^n$. Now for $\phi \in (C_c^\infty([0, t]))^d$, we have

$$\int_0^t v_i^n \dot{\phi} ds = - \int_0^t \dot{v}_i^n \phi ds = - \int_0^t \frac{1}{N} \sum_{k=1}^N (v_k^n - v_i^n) \psi_n(|x_i^n - x_k^n|) \phi ds$$

and the left-hand side converges to $\int_0^t v \dot{\phi} ds$. Thus it remains to show that the right-hand side converges to $[- \int_0^t \dot{v} \phi ds]$, where

$$\dot{v} := \frac{1}{N} \sum_{k=1}^N (v_k - v_i) \psi(|x_i - x_k|). \quad (2.12)$$

To this end we require for example that $\dot{v}^n \rightarrow \dot{v}$ in $(L^1([0, t]))^{Nd}$, which follows from Lemma 1.2 applied to functions $f_n = v_k^n - v_i^n$, $f = v_k - v_i$, $g_n = \psi(|x_i^n - x_k^n|)$, $g = \psi(|x_i - x_k|)$. \square

Remark 2.6. After arriving at (2.12) we may apply a stronger argument that $\dot{v}^n \rightarrow \dot{v}$ in $(L^1([0, t]))^{Nd}$. Clearly $x^n \rightarrow x$ and $v^n \rightarrow v$ a.e. and thus also $\dot{v}^n \rightarrow \dot{v}$ a.e.. Moreover $\dot{v}^n \in (L^1([0, t]))^{Nd}$ for all n . Therefore if we show that functions \dot{v}^n are uniformly integrable, then by Lemma 1.1 the proof will be finished. Since \dot{v}^n are uniformly bounded in $[\delta, t]$ for all $\delta > 0$ (as there are no collisions in $(0, T_0)^1$), then the set $\{0 \leq s \leq t : |\dot{v}^n| > c\}$ is included in some interval $[0, s_c]$ with $s_c \rightarrow 0$ as $c \rightarrow \infty$. Therefore by Lemma 2.1 we get

$$\int_0^t |\dot{v}^n| \chi_{\{s : |\dot{v}^n| > c\}} ds \leq \int_0^{s_c} |\dot{v}^n| ds \leq L |s_c|^{1-\alpha} \rightarrow 0$$

as $c \rightarrow \infty$, which proves that \dot{v}^n are uniformly integrable.

¹perhaps a better argument is that it follows from Lemma 2.1 and property 4 from Proposition 2.2

As our last effort in this section let us make an obvious remark involving properties stated in Proposition 2.2.

Corollary 2.2. *Properties 1,2,6,7 from Proposition 2.2 remain true also for the solution x on $[0, T_0)$. Moreover the following version of properties 3 and 4 holds:*

(4') *For all initial data $x(0)$ and $v(0)$ and all $i = 1, \dots, N$, we have*

$$\|v_i\|_{L^\infty([0, T_0])} \leq M,$$

where M is the constant from property 4 from Proposition 2.2.

Proof. Properties 1,2,4' follow by similar argumentation as in the proof of Proposition 2.2. Property 6 follows by definition of our system (namely by substituting equation (1.2) with (2.11)₂) and property 7 follows by calculating the derivative of

$$r(t) = \sum_{i,j} (v_i - v_j)^2.$$

□

Remark 2.7. Property 5 from Proposition 2.2 clearly does not hold on $[0, T_0)$ even though it holds on $[0, t]$ for $t \nearrow T_0$ (to see this, we simply substitute in property 5, n with $\lfloor \delta_t \rfloor + 2$, where δ_t is defined in part (A) of the proof of Proposition 2.3).

2.5 Clustering at the time of collision

In the previous section we established existence of solutions on the interval $[0, T_0)$, where T_0 is time of the first collision of some pair of particles. The solution x belongs to $(W^{2,1}([0, t]))^{Nd} \cap (C^1([0, T_0]))^{Nd} \cap (C([0, T_0]))^{Nd}$ for all $0 < t < T_0$ and satisfies (2.11) in a classical sense in $(0, T_0)$ and weakly in $(W^{2,1}([0, t]))^{Nd}$. Therefore we know that v is a Lipschitz continuous function in each interval $[0, t] \subset [0, T_0]$, however we do not know anything about its behaviour in a neighborhood of T_0 – with our current knowledge the limit of $v(t)$ as $t \rightarrow T_0$ may even not exist. In this section we provide a proof of continuity of v on whole interval $[0, T_0]$.

Definition 2.2. *For each $i, j = 1, \dots, N$ we define a relation $i \sim j$ if and only if $j \notin B_i(0)$ or for all $t < T_0$, we have*

$$\int_t^{T_0} \psi(|x_i - x_j|) ds = \infty.$$

This relation is clearly symmetric and reflexive but not necessarily transitive. This leads us to another definition.

Definition 2.3. For each $i, j = 1, \dots, N$ we define a relation \sim with the following two statements:

1. If $i \sim j$, then $i \sim j$.
2. For $i \not\sim j$, we have $i \sim j$ if and only if there exists k , such that $i \sim k$ and $k \sim j$.

Remark 2.8. Relation \sim is an equivalence relation. Since \sim is symmetric and reflexive then so is \sim . Transitivity of \sim follows directly from the definition. Equivalence classes $[i]$ of \sim provide us with a partition of the set of indexes $\{1, \dots, N\}$ with the following property: given $i, j = 1, \dots, N$ if $j \notin [i]$, then $\psi(|x_i - x_j|)$ is integrable in every interval $[t, T_0]$.

Now let us for each $i = 1, \dots, N$ define $w_i = w_i^{t_0}$ by the system of ODE's

$$\dot{w}_i = \frac{1}{N} \sum_{k \in [i]} (w_k - w_i) \psi(|x_i - x_k|)$$

in $[t_0, T_0)$ with the initial data $w_i(t_0) = v_i(t_0)$ for all $i = 1, \dots, N$. All structure based properties 1, 2 and 4' from Corollary 2.2 hold also for the functions w_i as in their proof we never make use of the fact that $\dot{x} = v$. We introduce the functions w_i as a tool to study the evolution of v in a neighborhood of T_0 . First we ensure that w_i and v_i are somehow close to each other and behave in a similar way.

Proposition 2.5. For $t \in [t_0, T_0)$, we have

$$|v_i(t) - w_i(t)| \leq \omega(T_0 - t_0),$$

for some nonnegative continuous function ω with $\omega(0) = 0$.

Proof. Let $r(t) = \sum_{i \in [i]} (v_i(t) - w_i(t))^2$. We have

$$r' = \frac{2}{N} \sum_{i, j \in [i]} (v_i - w_i) \left((v_j - v_i) - (w_j - w_i) \right) \psi(|x_i - x_j|) + \frac{2}{N} \sum_{i, j \notin [i]} (v_i - w_i)(v_j - v_i) \psi(|x_i - x_j|) =: I + II.$$

By the usual symmetry argument

$$\begin{aligned} I &= \frac{2}{N} \sum_{i, j \in [i]} \left((v_i - w_i)(v_j - w_j) - (v_i - w_i)^2 \right) \psi(|x_i - x_j|) = \\ &\quad - \frac{1}{N} \sum_{i, j \in [i]} \left((v_i - w_i) - (v_j - w_j) \right)^2 \psi(|x_i - x_j|) \leq 0. \end{aligned}$$

On the other hand II is integrable by Remark 2.8. Therefore, since $r(t_0) = 0$, for $t \in [t_0, T_0)$, we have

$$r(t) \leq \int_{t_0}^{T_0} |II| ds =: \omega^2(T_0 - t_0),$$

where ω is a nonnegative continuous function with $\omega(0) = 0$. □

Our next goal is to prove that if $i \sim j$ then $|w_i(t) - w_j(t)| \rightarrow 0$ as $t \rightarrow T_0$. However before we begin let us make another purely technical assumption that

$$\sum_{i \in [i]} w_i = 0. \quad (2.13)$$

This does not make our reasoning any less general since by property 2 from Corollary 2.2 this sum is constant in time – thus we may as well assume that it equals 0. Thus our goal can be rewritten in a equivalent form: prove that

$$\lim_{t \rightarrow T_0} w_i(t) = 0 \quad \text{for all } i \in [i]. \quad (2.14)$$

The first step of the proof is to show the following slightly weaker assertion.

Lemma 2.3. *If $i \sim j$, then there exists a sequence $s_n \rightarrow T_0$, such that $|w_i(s_n) - w_j(s_n)| \rightarrow 0$.*

Proof. The proof follows by contradiction. Let us assume that $i \sim j$ and there is no such sequence s_n i.e. there exists $\delta > 0$, such that $|w_i(s) - w_j(s)| > \delta$ for $s \in [t_0, T_0)$. Since $i \sim j$ both i and j belong to $[i]$ and thus for all $s \in [t_0, T_0)$ and for $r(s) := \sum_{k,l \in [i]} (w_k(s) - w_l(s))^2$, we have

$$r' = \frac{2}{N} \sum_{k,l,m \in [i]} (w_k - w_l) ((w_m - w_k)\psi(|x_k - x_m|) - (w_m - w_l)\psi(|x_l - x_m|)).$$

By a similar to the proof of property 3 from Proposition 2.2 application of the symetry we conclude that

$$r' = -2 \sum_{k,l \in [i]} (v_k - v_l)^2 \psi(|x_k - x_l|).$$

Now since $|w_i - w_j| > \delta$ and by property 4' from Corollary 2.2 also $\delta^2 < r(s) \leq NM^2$ and we have

$$(\ln r)' \leq -2 \frac{(v_i - v_j)^2}{r} \psi(|x_i - x_j|) \leq -\frac{2\delta^2}{NM^2} \psi(|x_i - x_j|)$$

and consequently

$$\delta^2 < r \leq e^{-\frac{2\delta^2}{NM^2} \int_{t_0}^s \psi(|x_i - x_j|) d\theta} r(t_0),$$

which is impossible since $\int_{t_0}^s \psi(|x_i - x_j|) \rightarrow \infty$ as $s \rightarrow T_0$. Therefore no such δ exists and the proof is complete. \square

Our next step is a technical lemma which is vaguely based on the fact that velocities of the particles only "pull" each other but never push away (which for example means that w_i which is the furthest from 0 may not go any further away from 0 because there is no other velocity to pull it there).

Lemma 2.4. For each $k = 1, \dots, d$ we denote w_i^k – the k -th coordinate of w_i and assume that up to permutations $w_1^k(t) \leq \dots \leq w_N^k(t)$. Then the sums

$$\sum_{i=1}^l w_i^k(t), \quad \text{and} \quad \sum_{i=l}^N w_i^k(t), \quad l = 1, \dots, N$$

are respectively nondecreasing and nonincreasing.

Proof. We prove the assertion only for the first sum as the other differs only by sign. For all $l = 1, \dots, N$, we have

$$\left(\sum_{i=1}^l w_i^k \right)' = \sum_{i,j=1}^l (w_j^k - w_i^k) \psi(|x_i - x_j|) + \sum_{i=1}^l \sum_{j=l+1}^N (w_j^k - w_i^k) \psi(|x_i - x_j|) =: I + II.$$

By symmetry $I = 0$. On the other hand for $j > l$ as long as $w_j^k - w_i^k > 0$, we have $II \geq 0$ and the sum $\sum_{i=1}^l w_i^k$ is nondecreasing. \square

Now we may proceed with our goal which is the following proposition.

Proposition 2.6. If $i \sim j$ then

$$\lim_{t \rightarrow T_0} |w_i(t) - w_j(t)| = 0. \quad (2.15)$$

Proof. It suffices to show that the assertion holds if we substitute w_i with w_i^k – it's k -th coordinate, thus let us assume for simplicity of notation that $w_i = w_i^k$. Therefore w_i are real functions and their sum equals to 0 by (2.13). The proof follows by 3 steps.

STEP 1. For $t \in [t_0, T_0)$, let

$$\mathcal{R}(t) := \max_{j \in [l]} w_j(t).$$

First we prove that if at some point $t \in [t_0, T_0)$ we have

$$w_i(t) = \mathcal{R}(t) - \delta, \quad (2.16)$$

then

$$\sup_{s \in [t, T_0)} w_i \leq \mathcal{R}(t) - \frac{\delta}{N!}. \quad (2.17)$$

This means that if some velocity w_i is far away from the highest velocity at the time t , then the highest possible value for w_i is significantly smaller than the highest velocity at the time t . The proof follows by induction with respect to the number of velocities

w_j that are bigger than w_i at the time t . For $n = 1$ we are in a situation when there is only one w_j , such that $\mathcal{R}(t) = w_j(t) > w_i(t)$ and (2.16) implies that $\mathcal{R}(t) - w_i(t) = \delta$. Now let

$$p(s) := \max\{w_k(s) : w_k(s) < \mathcal{R}(s)\}, \quad \text{for } s \in [t, T_0)$$

Clearly $p(t) = w_i(t)$ but it is possible that some other velocity may become bigger than w_i at some point in time and this is the only reason to introduce the function p , which will serve us by pointing the right-hand edge of the set of velocities smaller than \mathcal{R} . Clearly $w_i \leq p \leq \mathcal{R}$ in $[t, T_0)$. Moreover Lemma 2.4 implies that the sum $p + \mathcal{R}$ is nonincreasing. Therefore

$$\mathcal{R}(t) \geq \mathcal{R}(s) + p(s) - p(t) \geq 2w_i(s) - w_i(t) = 2w_i(s) - \mathcal{R}(t) + \delta,$$

which implies that

$$\sup_{s \in [t, T_0)} w_i \leq \mathcal{R}(t) - \frac{\delta}{2}.$$

Now let us assume that condition (2.16) implies that

$$\sup_{s \in [t, T_0)} w_i \leq \mathcal{R}(t) - \frac{\delta}{(n+1)!} \quad (2.18)$$

in case when at the time t there are exactly n velocities bigger than w_i . We will prove that this implies that if (2.16) holds, then

$$\sup_{s \in [t, T_0)} w_i \leq \mathcal{R}(t) - \frac{\delta}{(n+2)!} \quad (2.19)$$

if only there are exactly $n+1$ velocities bigger than w_i at the time t . In such case we define

$$p(s) := \max\{w_k(s) : k \notin \mathcal{G}\}, \quad \text{for } s \in [t, T_0),$$

where \mathcal{G} is the set of indexes of the $n+1$ biggest velocities at the time t . Here again the sole purpose of the function p is to point the biggest velocity that was initially smaller than the biggest $n+1$ velocities. Denoting $\sum_{k \in \mathcal{G}} w_k(s) =: S(s)$, by Lemma 2.4, the function $S + p$ is nonincreasing as long as

$$p(s) < \min_{k \in \mathcal{G}} w_k(s), \quad (2.20)$$

thus

$$(n+2)p(s) < S(s) + p(s) \leq S(t) + p(t) = S(t) + \mathcal{R}(t) - \delta \leq (n+2)\mathcal{R}(t) - \delta,$$

$$p(s) < \mathcal{R}(t) - \frac{\delta}{n+2}$$

as long as (2.20) holds. So if at some time s_0 , we have $p(s_0) = \mathcal{R}(t) - \frac{d}{n+2}$ then also $p(s_0) \geq \min_{k \in \mathcal{G}} w_k(s_0)$. At that point there are at most n velocities bigger than p and the distance between $p(s_0)$ and $\mathcal{R}(s_0)$ is no less than $\delta' := \frac{\delta}{n+2}$. Therefore by (2.18), we have

$$p(s) \leq \mathcal{R}(s_0) - \frac{\delta'}{(n+1)!} \leq \mathcal{R}(t) - \frac{\delta}{(n+2)!} \quad \text{for } s \in [s_0, T_0).$$

This proves (2.19). Noticing that $n \leq N-1$ we get (2.17) and finish step 1.

STEP 2. Our next step is the following simple observation with the proof left for the reader.

Lemma 2.5. *If (2.15) does not hold, then there exists $\epsilon > 0$ and a sequence $s_n \rightarrow T_0$, such that for all $i \in [i]$ there exists $j \in [i]$, such that*

$$|w_i(s_{n_k}) - w_j(s_{n_k})| \rightarrow 0 \quad \text{and} \quad |w_i(s_{n_l}) - w_j(s_{n_l})| > \epsilon$$

for some subsequences $\{s_{n_k}\}, \{s_{n_l}\} \subset \{s_n\}$.

STEP 3. Now we finish the prove. Let us fix $t \in [t_0, T_0)$ and assume that w_i is one of the biggest velocities at the time t i.e. $\mathcal{R}(t) = w_i(t)$. Lemma 2.5 ensures existence of j , such that

$$|w_i - w_j| \rightarrow 0 \tag{2.21}$$

on one subsequence converging to T_0 and

$$|w_i - w_j| > \epsilon \tag{2.22}$$

on some other subsequence converging to T_0 for ϵ independent of i and j . Thus (2.22) implies that at some time $s \in [t, T_0)$ either w_i or w_j (say w_j) is farther from $\mathcal{R}(t)$ than ϵ . Then step 1 implies that

$$\sup_{\theta \in [s, T_0)} w_j \leq \mathcal{R}(t) - \frac{\epsilon}{N!}.$$

Moreover (2.21) implies that at some other time $r \in [s, T_0)$, we have

$$w_i(r) \leq \mathcal{R}(t) - \frac{\epsilon}{2N!}$$

and after that point (again by step 1)

$$\sup_{\theta \in [r, T_0)} w_i \leq \mathcal{R}(t) - \frac{\epsilon}{(2N!)^2}.$$

This procedure can be performed with any velocity w_i that at some time equals to \mathcal{R} as many times as we want. Therefore we may make sure that $\mathcal{R}(t)$ is arbitrarily small at some time $t < T_0$. The same can be done with $\mathcal{L}(t) := \min_{j \in [i]} w_j(t)$ to conclude that diameter of velocities converges to 0 as $t \rightarrow T_0$ and this contradicts (2.22) and by Lemma 2.5 implies that assertion (2.15) is true. This finishes the proof. \square

Remark 2.9. In Proposition 2.6 we proved that for all $i \in [i]$ we have $w_i \rightarrow 0$. However this was under our assumption (2.13). Now it is time to drop this assumption and conclude that in general there exists a constant \bar{v} , such that for all $i \in [i]$, we have $w_i \rightarrow \bar{v}$.

Our last goal in this subsection is to clarify what does Proposition 2.6 imply to the motion of v .

Corollary 2.3. *There exists a constant \bar{v} , such that for all $i \in [i]$, we have*

$$\lim_{t \rightarrow T_0^-} v_i(t) = \bar{v}.$$

Proof. Given $\epsilon > 0$ we need to ensure existence of $s_0 < T_0$, such that for all $s_0 < s < T_0$, we have

$$|v_i(s) - \bar{v}| < \epsilon$$

Let t_0 be such that $\omega(T_0 - t_0) < \frac{\epsilon}{2}$, where t_0 and ω are as in Proposition 2.5. By Remark 2.9 there exists $s_0 \in [t_0, T_0)$, such that for all $s \in [s_0, T_0)$, we have

$$|v_i(s) - \bar{v}| \leq |v_i(s) - w_i^{t_0}(s)| + |w_i^{t_0}(s) - \bar{v}| < \omega(T_0 - t_0) + \frac{\epsilon}{2} < \epsilon.$$

□

This finally proves that the function v has a limit at T_0^- and we may extend it continuously to $[0, T_0]$. For the sake of clarity of argumentation in the next section, it is useful to summarise what we proved in this section.

Remark 2.10. We actually proved that

1. If

$$\int_t^{T_0} \psi(|x_i - x_j|) ds = \infty$$

for all $t < T_0$, then $v_i(t) - v_j(t) \rightarrow 0$ as $t \rightarrow T_0$

2. If on the other hand v_i is separated from v_j in a left sided neighborhood of T_0 then we have

$$\int_t^{T_0} \psi(|x_i - x_j|) ds < \infty \tag{2.23}$$

3. Condition (2.23) holding for all i and $j \in B_i(0)$ implies that all functions v_i are in fact weak solutions of (2.11)₂ in $W^{1,1}([0, T_0])$ and in particular admit a modulus of continuity and are uniformly continuous at T_0^- . Therefore again there exists a limit of v_i at T_0 but it does not necessarily equal to a limit of v_j for any $j \in B_i(0)$.

2.6 Global existence

In this section we combine our efforts from sections 2.4 and 2.5 to obtain global existence in the sense of Definition 2.1. Propositions 2.3 and 2.4 ensure existence of weak solutions in $[0, T_0)$ with a continuous velocity v . Corollary 2.3 implies that there exists a left sided limit of v at T_0 . Thus it may be extended continuously to $[0, T_0]$. Therefore for arbitrary initial data there exists a unique solution $x \in C^1([0, T_0])^{dN}$ satisfying Definition 2.1. Now assuming that T_0 is a new initial point with initial data equal to $x(T_0)$ and $v(T_0)$ and applying Proposition 2.3 we conclude that the solution exists on $[0, T_n]$, where T_n is $n + 1$ -th time at which some particles collide. What is not clear however is whether for arbitrary T we may find $T_n \geq T$.

Proposition 2.7. *For all $T > 0$ there exists a $(C^1([0, T]))^{Nd}$ solution of (1.1) with arbitrary initial data. This solution is in the sense of Definition 2.1.*

Proof. It suffices to show that we may extend our solution up to an arbitrary $T > 0$. Let T_n be a sequence of the points of collision and assume by contradiction that $\sum_n (T_n - T_{n-1}) < \infty$. Then $T_n - T_{n-1} \rightarrow 0$ and $T_n \rightarrow \tilde{T}$ for some $\tilde{T} > 0$. We will prove that \tilde{T} is a point of sticking for some particles x_i and x_j . Clearly there exist i and $j \in B_i(0)$ and a subsequence T_{n_k} such that $x_i(T_{n_k}) - x_j(T_{n_k}) = 0$, which by Lipschitz continuity of x (property 4' from Corollary 2.2) implies that $x_i(t) - x_j(t) \rightarrow 0$ as $t \rightarrow \tilde{T}$ and \tilde{T} is a point of collision of x_i and x_j . Now it remains to show that $v_i(t) - v_j(t) \rightarrow 0$ as $t \rightarrow \tilde{T}$. If

$$\int_t^{\tilde{T}} \psi(|x_i - x_j|) ds = \infty \quad (2.24)$$

for all $t < \tilde{T}$, then by Remark 2.10,1 we are done. On the other hand if $\psi(|x_i - x_j|)$ is integrable in a left sided neighborhood of \tilde{T} then by Remark 2.10,3, velocity v is uniformly continuous at \tilde{T}^- and in particular has a limit at \tilde{T} . Therefore there exists a limit of $v_i - v_j$ at \tilde{T} . If this limit equals to 0 then, again, we are done. If on the other hand it equals to some $\xi \neq 0$, then in a neighborhood of \tilde{T} we have $v_i - v_j \in B(\xi, \epsilon)$, where $B(\xi, \epsilon)$ is a ball centered at ξ with an arbitrary small radius ϵ . This implies a clearly false statement that

$$0 = x_i(T_{n_{k+1}}) - x_j(T_{n_{k+1}}) \in (T_{n_{k+1}} - T_{n_k})B(\xi, \epsilon)$$

with T_{n_k} and $T_{n_{k+1}}$ sufficiently close to \tilde{T} . This contradicts the assumption that $\xi \neq 0$. Altogether we proved that if \tilde{T} is a density point for T_n then it is a point of sticking of the particles. Then we may further extend our solution beyond \tilde{T} . Finally, since there can be at most $N - 1$ times of sticking, then for all $T > 0$ either we can find T_n such that $T_n > T$ or all the particles stick together before time T and travel with constant velocity for as long as we want them to. \square

3 On the case of two particles – flocking in a finite time

In this section our goal is to discuss the possibility of a finite in time alignment in case of two particles ($N = 2$). First let us recall that asymptotic flocking was studied before in most papers mentioned in the introduction, see e.g. [26] and we refer to those papers to see general definitions and results. Here, we consider the most strict form of flocking, which is sticking of the trajectories of the particles in a finite time. By property 2 from Corollary 2.2 the average velocity of the particles is constant, which means that

$$v_1 \equiv -v_2 + \bar{v}$$

for some constant \bar{v} . Without a loss of generality we may assume that $\bar{v} = 0$. The above observation implies that

$$x_1(t) = -x_2(t) + t\bar{v} + (x_1(0) + x_2(0))$$

and assuming without a loss of generality that also $x_1(0) = -x_2(0)$, we get $x_1 \equiv -x_2$. Thus both, average velocity and the center of mass of the particles are equal to 0. Therefore the particles move parallelly to each other, either on two separate parallel lines or on the same line. In the former case, the distance between particles is always no less than the distance of respective lines, thus there is no possibility of a finite in time (or asymptotic for that matter) alignment. In the latter case the distance between particles can be arbitrarily small, thus hypothetically a finite in time alignment may occur. In order to simplify our calculations, since particles move on the same line, then by a simple change of variables we may assume that $d = 1$. Altogether we have two particles x_1 and x_2 , with $x_1 \equiv -x_2$ and $v_1 \equiv -v_2$. Therefore they are unequivocally defined by the function

$$\phi(t) := x_2(t) - x_1(t).$$

Then the C-S model (2.11) (or (1.1) and (1.2), since in this case they are the same) can be rewritten equivalently as

$$\ddot{\phi}(t) = -2\dot{\phi}(t)\psi(|\phi(t)|), \tag{3.1}$$

with $\phi(0) = x_2(0) - x_1(0) \geq 0$ and $\dot{\phi}(0) = v_2(0) - v_1(0) \in \mathbb{R}$. Moreover Lemma 2.4 implies that if at some time t we have $\dot{\phi}(t) = 0$ then it will be constantly equal to 0 from that point in time. This implies that $\dot{\phi}$ may not change sign and this further implies that there may be at most one collision of the particles. Finally let us notice that by Proposition 2.7 there exists a solution to (3.1) with arbitrary initial data and we can easily prove that if $\phi(0) > 0$, then this solution is unique. Now we are ready to state our main result of this section.

Proposition 3.1. *Let ϕ be a solution of (3.1) with $\phi(0) > 0$. Then the following are equivalent:*

1. *There exists a time $t_0 < \infty$ such that $\phi(t_0) = \dot{\phi}(t_0) = 0$.*
2. *Initial data satisfy:*

$$\dot{\phi}(0) = -2\Psi(\phi(0)), \quad (3.2)$$

where $\Psi(s) := \frac{1}{1-\alpha}s^{1-\alpha}$ is a primitive of ψ .

Proof. Since there is at most one collision of the particles and we know that they stick together, thus ϕ and $\dot{\phi}$ have constant signs. Therefore since $\phi(0) > 0$ then also $|\phi| = \phi$ and by simple integration of (3.1) we conclude that the function ϕ satisfies:

$$\dot{\phi}(t) = -2\Psi(\phi(t)) + 2\Psi(\phi(0)) + \dot{\phi}(0) \quad (3.3)$$

and

$$\phi(t) = -2 \int_0^t \Psi(\phi(s))ds + t(2\Psi(\phi(0)) + \dot{\phi}(0)) + \phi(0). \quad (3.4)$$

Substituing t with t_0 in (3.3) we obtain

$$0 = 2\Psi(\phi(0)) + \dot{\phi}(0),$$

which is exactly condition (3.2). Now let us assume that (3.2) is satisfied. We are going to prove existence of t_0 . First note that in our case (3.3) and (3.4) are satisfied on the set $\{t : \phi(t) \geq 0\}$ and they have the following form:

$$\begin{aligned} \dot{\phi}(t) &= -2\Psi(\phi(t)), \\ \phi(t) &= -2 \int_0^t \Psi(\phi(s))ds + \phi(0). \end{aligned} \quad (3.5)$$

From (3.5) and by the definition of ψ we obtain

$$\dot{\phi}(t) = -\frac{2}{1-\alpha}\phi(t)\psi(\phi(t))$$

and

$$\phi(t) = e^{-\frac{2}{1-\alpha} \int_0^t \psi(\phi(s))ds} \phi(0).$$

Thus, since $\max_{t \in [0, t_0]} \phi(t) = \phi(0)$, we have

$$\phi(t) \leq e^{-\frac{2}{1-\alpha} t \psi(\phi(0))} \phi(0),$$

which can become arbitrarily small in a finite time. Now for $n = 2, 3, \dots$ let

$$t_n := \inf\{t > t_{n-1} : \phi(t) \leq 2^{-n}\},$$

with $t_1 := 0$. We have

$$\begin{aligned}\phi(t_n) &= e^{-\frac{2}{1-\alpha} \int_{t_{n-1}}^{t_n} \psi(\phi(s)) ds} \phi(t_{n-1}), \\ 2^{-1} &= e^{-\frac{2}{1-\alpha} \int_{t_{n-1}}^{t_n} \psi(\phi(s)) ds}, \\ \ln 2 &= \frac{2}{1-\alpha} \int_{t_{n-1}}^{t_n} \psi(\phi(s)) ds \geq \frac{2}{1-\alpha} (t_n - t_{n-1}) 2^{\alpha(n-1)}.\end{aligned}$$

Therefore

$$(t_n - t_{n-1}) \leq \frac{(1-\alpha) \ln 2}{2} 2^{\alpha(1-n)}$$

and t_n is a partial sum of a convergent series. Thus t_n converges to a finite limit t_0 such that $\phi(t_0) = \dot{\phi}(t_0) = 0$. \square

Remark 3.1. Finally let us mention that a finite in time allignment may not happen in case of ψ_{cs} defined by (1.4) since (3.1) implies that

$$|\dot{\phi}(t)| = e^{-2 \int_0^t \psi_{cs}(|\phi(s)|) ds} \dot{\phi}(0) \geq e^{-2t \|\psi_{cs}\|_\infty} |\dot{\phi}(0)| > 0,$$

as long as $\dot{\phi}(0) \neq 0$. Moreover we may just as easily prove that with unintegrable singular weight ψ , e.g. when $\psi(s) = s^{-\alpha}$ for $\alpha > 1$ in one dimensional setting not only particles cannot stick but they cannot even collide.

Conditions described in Proposition 3.1 refer to the function ϕ and in a simplified case of one dimension. However they can be modified to cover more general cases and refer directly to x_1 and x_2 .

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